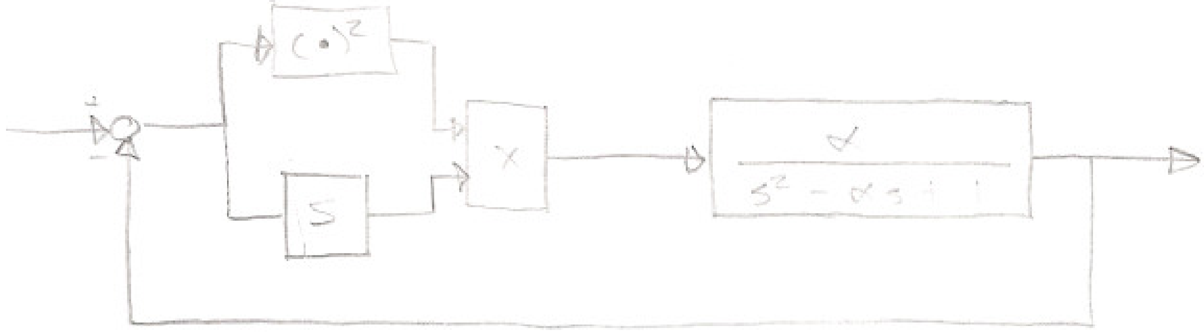
Ex: Van der Pol equation

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

α is a positive parameter

$$\ddot{x} - \alpha\dot{x} + x = -\alpha x^2 \dot{x}$$



The linear block, although ustable, has low pass properties. Let's assume that the system has a limit cycle with frequency ω and amplitude A.

$$x(t) = A \sin \omega t$$

$$\dot{x} = A\omega \cos \omega t$$

$$w = -x^2\dot{x} = -A^2 \sin^2 \omega t \, A\omega \cos \omega t$$

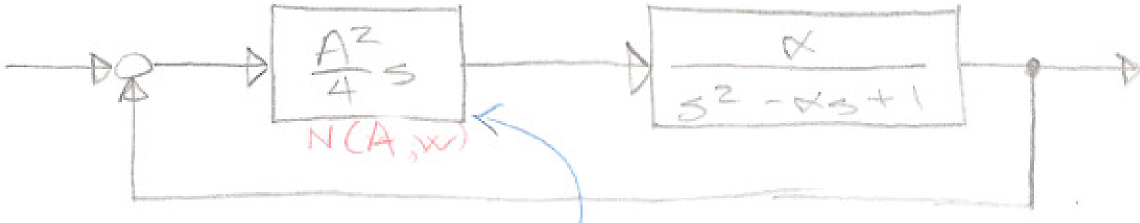
$$= -\frac{A^3\omega}{4}\left[\cos \omega t - \cos 3\omega t\right]$$

Since the linear block has low pass properties, we can reasonably assume that the 3rd harmonic ($\cos 3\omega t$) term is sufficently attinuated by the linear block and its effect is not present in the signal flow after the linear block.

This means that

$$w \approx -\frac{A^3\omega}{4}\cos \omega t = \frac{A}{4}\frac{d}{dt}\left[-A \sin \omega t\right]$$

$$= \frac{A^2}{4}\frac{d}{dt}\left[-x\right]$$



describing func.
approx. of non. linear block. (considering just first harmonic)

$$w = N(A,w)(-x)$$

$$N(A,w) = \frac{A^2}{4}(j\omega) \qquad \boxed{s = j\omega}$$

$$x = A\sin\omega t = G(j\omega)w = G(j\omega)N(A,w)(-x)$$

$$G(s) = \frac{\alpha}{s^2 - \alpha s + 1}, \quad G(j\omega) = \frac{\alpha}{(j\omega)^2 - \alpha(j\omega) + 1}$$

$$x = G(j\omega)N(A,w)(-x)$$

$$\boxed{1 + G(j\omega)N(A,w) = 0}$$

characteristic Equation

$$\boxed{1 + \frac{A^2}{4}(j\omega)\frac{\alpha}{(j\omega)^2 - \alpha(j\omega) + 1}} \Rightarrow \begin{array}{l} A = 2 \\ \omega = 1 \end{array}$$

note: The first harmonic of this system has a freq of 1 and amplitude of 2.

we can go farther to see if the limit cycle is stable, unstable, or semistable. We look to the characteristic equation

$$1 + \frac{A^2}{4} s \frac{\alpha}{s^2 - \alpha s + 1}$$

the closed loop poles are

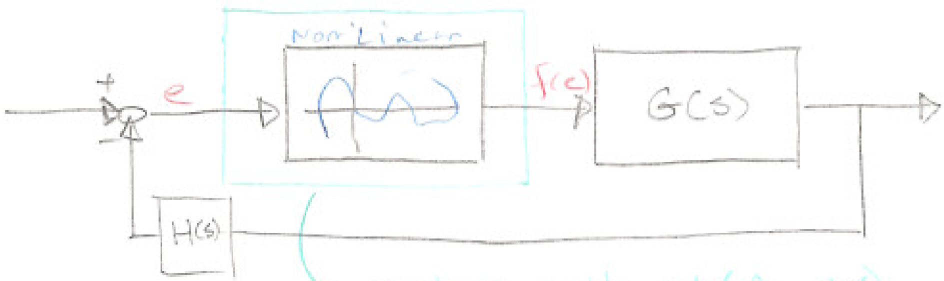
$$P_{1,2} = -\frac{1}{8}\alpha(A^2-4) \pm \sqrt{\frac{1}{64}\alpha^2(A^2-4)^2 - 1}$$

$A < 2$ then the system is unstable.

$A > 2$ the the system is stable

conclusion, the limit cycle is stable.

most non linear systems have a format.



replace with $N(A, \omega)$

Assume $e = A\sin\omega t$

Let the fundimental terms in the fourier expansion of $f(e)$ be

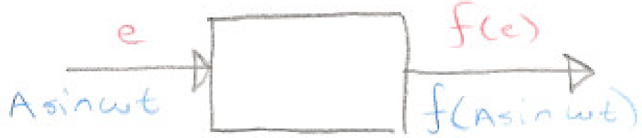
$$b_1 \cos\omega t + a_1 \sin\omega t =$$

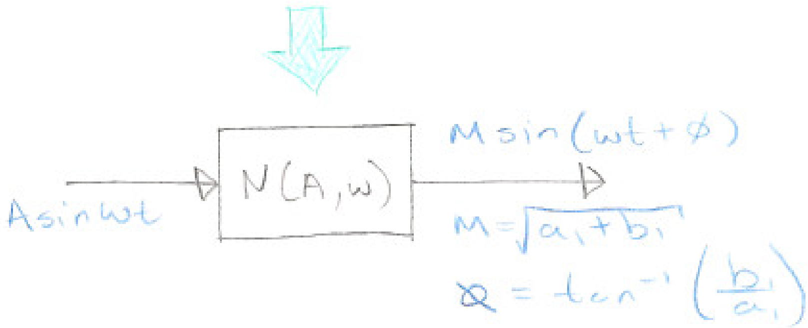
$$= \sqrt{a_1^2 + b_1^2}\,\sin(\omega t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

The describing function is:

$$N(A,\omega) = \sqrt{\frac{a_1^2 + b_1^2}{A}}\, e^{j\phi}$$





Fourier expansion, a periodic func. $w(t)$ can be.

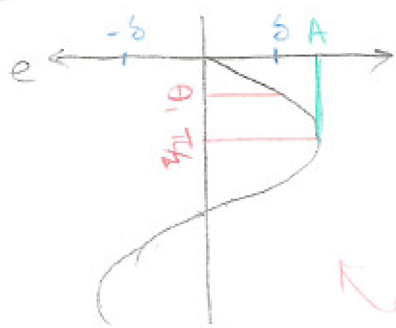
$$w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \sin(n\omega t) + b_n \cos(n\omega t)\right]$$

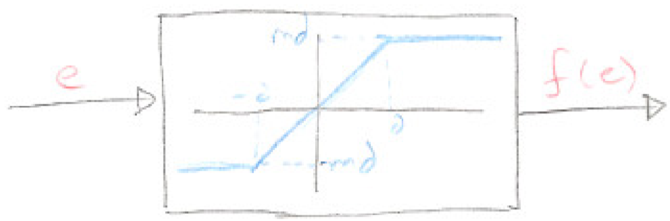
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\, d(\omega t)$$

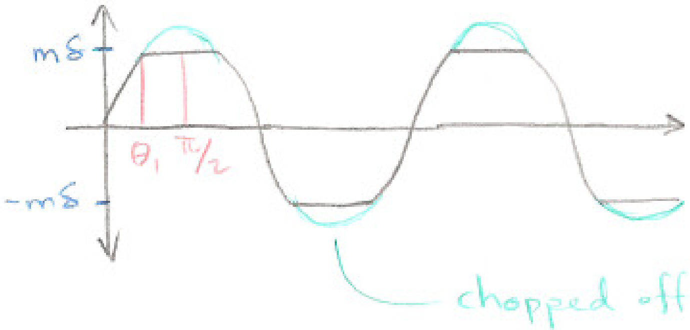
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t) \sin(n\omega t)\, d(\omega t)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t) \cos(n\omega t)\, d(\omega t)$$

EX: Saturation.







$$e = A \sin \omega t$$

$$f(e) = f(A \sin \omega t) = \begin{cases} mA \sin \theta & \text{if } 0 \le \theta \le 1 \\ m\delta & \text{if } \theta_1 < \theta < \pi/2 \end{cases}$$

$$A \sin \theta = \delta \Rightarrow \theta_1 = \sin^{-1} \delta$$